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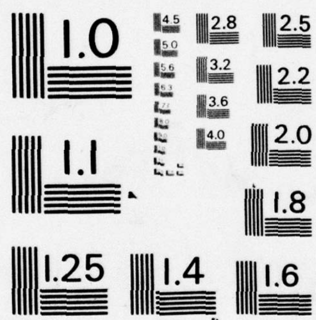

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ON THE COMPUTATION OF SINGULAR CONTROLS*

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ABSTRACT

We consider singular optimal control problems consisting of a state equation

$$\dot{x} = Ax + Bu$$

for vectors x and scalars u and a cost functional

$$J = \frac{1}{2}\int_0^T (x'Qx + \epsilon^2 u^2)dt$$

to be minimized for $|u| \leq m$ and $\epsilon = 0$. By considering the problem as $\epsilon \to 0$, singular perturbation concepts can be used to compute solutions consisting of bang-bang controls followed by singular arcs. The procedure further develops a numerical technique proposed by Jacobson, Gershwin, and Lele, as well as additional analytic methods developed by other authors.

1. INTRODUCTION

A typical singular optimal control problem consists of a state equation

$$\dot{x} = Ax + bu, \quad 0 \leq t \leq T < \infty \tag{1.1}$$

(subject to end conditions on the n-vector x) and a scalar cost functional

$$J = \frac{1}{2}\int_0^T x'Q\,x\,dt \tag{1.2}$$

which is to be minimized for a symmetric, positive semi-definite matrix Q (i.e., $Q \geq 0$) and for a scalar control u which is restricted to lie within the finite bounds

$$-m \leq u \leq m. \tag{1.3}$$

The known theory of singular problems has recently been summarized in Bell and Jacobson (1975), while research papers on the general problem and particular applications abound in the current literature.

Jacobson and coworkers (cf. Jacobson and Speyer (1971) and Jacobson, Gershwin, and Lele (1970)) converted such problems to nonsingular (though nearly singular) problems by considering the perturbed cost

$$J(\epsilon) = \frac{1}{2}\int_0^T (x'Qx + \epsilon^2 u^2)dt \tag{1.4}$$

for a sequence of positive ϵ values tending toward zero. This device resulted in considerable progress both for theoretical purposes and for numerical calculation. More recently, O'Malley and Jameson (1975, 1976) and O'Malley (1976) have used an analogous method to analytically solve such singular control problems with time-varying coefficients and vector controls, but without bounds on the components of the control. Their results rely heavily on the asymptotic theory for singularly perturbed boundary value problems for ordinary differential equations (cf. Wasow (1965), Lions (1973), and O'Malley (1974)). Applications of such analyses in control have recently been surveyed by Kokotovic et al. (1976). Bell and Jacobson (1975) stress the need and difficulty of developing computational techniques for singular control problems and observe that the best current schemes are the "ϵ-method" of Jacobson et al. (1970) and gradient techniques (cf., e.g., Pagurek and Woodside (1968), Sirisena (1974), and Edge and Powers (1976)). We note that these ϵ-methods are analogous to the artificial viscosity methods popular in computational fluid dynamics (cf. Richtmyer and Morton (1967)). In addition, we note that Boggs (1976) has overcome some difficulties with gradient techniques by using asymptotic methods.

Jacobson et al. (1970) and O'Malley and Jameson (1975) show that although the optimal cost $J(\epsilon)$ given in (1.4) converges uniformly as $\epsilon \to 0$, the limiting control generally converges nonuniformly. Indeed, the control must be expected to consist of bang-bang arcs and singular arcs (cf. Johnson and Gibson (1963)) and convergence is necessarily nonuniform as $\epsilon \to 0$ at switch points. [In another context, we note that the natural sluggish transfer of mechanical systems at switch points of bang-bang control could be modelled by another singular perturbation device, viz. the introduction of a cost term $\int_0^T \mu^2(\dot{u}(t))^2 dt$ for μ small.] We also recall that Powers and McDanell (1971) and Edgar and Lapidus (1972) report practical success in using the ϵ-method for Saturn rocket guidance and chemical reactor problems. Practical difficulties certainly remain, however. As $\epsilon \to 0$, for example, Jacobson et al. (1970) note that numerical instability manifests itself and they suggest a sufficiently good approximation might result by reducing ϵ to a small, but "still numerically stable" value. Analogous difficulties have, of course, been common in the numerical solution of boundary value problems for stiff ordinary differential equations (cf. Willoughby (1974) or Aziz (1975)). Substantial progress has been made on these problems by using singular perturbation ideas to develop specially-tailored numerical methods (cf. Miranker (1973), Ferguson (1975), Flaherty and O'Malley (1977), and Kreiss and Nichols (19)). We propose a similar program to further study singular control problems. By combining asymptotic and numerical ideas, improved methods will necessarily follow. We cannot, of course, obliterate the unavoidable complicated behavior inherent in these singular problems (cf.,e.g., Fuller (1963)). We observe that the nearly singular problems are likely to be of independent interest (cf. O'Malley and Jameson (1975)), though we shall not pursue that question here.

In their recent study of singular arc problems with unbounded controls, Jameson and O'Malley found considerably different behavior in a sequence of cases. Case k, $k = 1, 2, \ldots,$ corresponds to the more familiar classification of singular arcs of order k (cf. Robbins (1967) or Goh (1966)) and is defined such that $b'(A')^j QA^j b = 0$ for $j = 0, 1, \ldots, k-2$ and $b'(A')^{k-1} QA^{k-1} b > 0$. A given problem need not fit into any such case (as when $Q = 0$) and for vector controls may lie between cases (when the last matrix is singular, but of positive rank (cf. Anderson (1973)). Like much of the literature (cf. Wonham and Johnson (1964), Sirisena (1968), and Anderson (1972)), our discussion will mostly concern Case 1. For Case k problems, the singular arc solution for an n-vector x and scalar u involves a control law for a dynamical system of order $n - k$ (cf. O'Malley and Jameson (1976)). It is natural then to seek a control u which switches $k - 1$ times between control bounds $\pm m$ before reaching the singular arc. It is easy to see that such a control will not generally be optimal for $k \geq 2$, though we show that it is quite satisfactory for certain examples. We must anticipate such difficulties, however, since experts conjecture that infinite switching (as in the classical Fuller problem) is generic for $k \geq 2$ (cf. Krener (1976)). We note that the nonsingular problem with a small fixed value of ϵ could have an optimal solution with a finite number of switchings, while the limiting singular problem involves infinite switchings.

2. TWO FIRST ORDER SINGULAR ARC PROBLEMS

a. The simplest singular arc problem may be

$$\begin{cases} \dot{x} = u, \quad x(0) = 1 \\ \text{with} \\ J = \frac{1}{2}\int_0^2 x^2(t)dt \\ \text{to be minimized for } |u| \le m. \end{cases}$$

For $m = \infty$, the optimal control features an initial negative delta function impulse which drives the state immediately to zero with the optimal cost $J^* = 0$. *To solve this problem by the* ϵ*-method we consider the nonsingular problem*

$$\begin{cases} \dot{x} = u, \quad x(0) = 1 \\ \text{with} \\ J(\epsilon) = \frac{1}{2}\int_0^2 (x^2(t) + \epsilon^2 u^2(t))dt \\ \text{to be minimized.} \end{cases}$$

Its unique solution for $m = \infty$,

$$\begin{cases} x(t,\epsilon) = (1 + e^{-4/\epsilon})^{-1}(e^{-t/\epsilon} + e^{-2/\epsilon}e^{-(2-t)/\epsilon}) \\ u(t,\epsilon) = -\frac{1}{\epsilon}(1 + e^{-4/\epsilon})^{-1}(e^{-t/\epsilon} - e^{-2/\epsilon}e^{-(2-t)/\epsilon}), \end{cases}$$

has the asymptotic limit

$$x(t,\epsilon) \sim e^{-t/\epsilon} \quad \text{and} \quad u(t,\epsilon) \sim -\frac{1}{\epsilon}e^{-t/\epsilon}$$

for $\epsilon \to 0^+$. *We note, in particular, that this limiting control behaves like* $-\delta$ at $t = 0$ since

$$\frac{1}{\epsilon}\int_0^2 f(t)e^{-t/\epsilon}dt \to f(0) \quad \text{as} \quad \epsilon \to 0.$$

For bounded controls, $|u| \leq m$, the optimal control does the best it can (cf. e.g., Sage (1968)), viz. for $m > \frac{1}{2}$

$$u = \begin{cases} -m, & 0 \leq t < \frac{1}{m} \\ 0, & t > \frac{1}{m}. \end{cases}$$

The integrated effect is the same, i.e., the singular arc solution $x = u = 0$ is ultimately reached (see Figure 1). Indeed, the preceding results can be recovered by letting $m \to \infty$. For bounded m, the problem can be explicitly solved by using the cost functional $J(\epsilon)$ and seeking the asymptotic solution as $\epsilon \to 0$. For $m = 1$, the limiting cost is exactly $\frac{1}{6}$ compared to Jacobson et al. (1970)'s calculated values 0.1717 and 0.1617.

Slight modifications of this example can be easily handled, e.g., Rozonoer's example

$$\dot{x} = u, \quad x(0) = 1, \quad x(2) = \frac{1}{2}, \quad |u| \leq 1, \quad J = \frac{1}{2}\int_0^2 x^2 dt$$

(cf. Pagurek and Woodside (1968)) and the vector control problem

$$\dot{x}_i = iu_i, \quad |u_i| \leq 1, \quad i = 1, 2, \quad J = \frac{1}{2}\int_0^2 (x_1^2 + x_2^2) dt.$$

b. As a second example, consider the harmonic oscillator problem

$$(2.1) \qquad \begin{cases} \dot{x}_1 = x_2, \quad x_1(0) = 0 \\ \dot{x}_2 = u, \quad x_2(0) = 1 \\ \text{with cost} \\ J(\epsilon) = \frac{1}{2}\int_0^5 (x_1^2 + x_2^2 + \epsilon^2 u^2) dt \\ \text{and } |u| \leq m. \end{cases}$$

For $m = \infty$, the usual state-costate formulation (cf., e.g., Athans and

Falb (1965)) implies that the optimal control will be given by

$$u = -p_2/\epsilon^2 \tag{2.2}$$

where the state and costate vectors $x = (x_1, x_2)'$ and $p = (p_1, p_2)'$ satisfy

$$\begin{aligned} &\dot{x}_1 = x_2, \quad x_1(0) = 0; \qquad \dot{x}_2 = u, \quad x_2(0) = 1 \\ &\dot{p}_1 = -x_1, \quad p_1(5) = 0; \qquad \dot{p}_2 = -x_2 - p_1, \quad p_2(5) = 0. \end{aligned} \tag{2.3}$$

The asymptotic solution of the singularly perturbed two point boundary value problem (2.2)-(2.3) can be readily obtained. It consists of an initial boundary layer (endpoint region of nonuniform convergence) while the limiting solution within $(0,5)$ satisfies the limiting problem

$$\dot{X}_{10} = X_{20}, \quad 0 = \rho_{20}, \quad \dot{\rho}_{10} = -X_{10}, \quad \text{and} \quad \dot{\rho}_{20} = -X_{20} - \rho_{10}$$

obtained when we set $\epsilon = 0$. Since $\rho_{20} = 0$ and $X_{20} = -\rho_{10}$, we're left with a linear system for X_{10} and ρ_{10}. If we now use the boundary values $X_{10}(0) = x_1(0) = \rho_{10}(5) = P_1(5) = 0$, we get the trivial singular arc solution

$$X_{10} = \rho_{10} = X_{20} = \rho_{20} = 0. \tag{2.4}$$

Further calculations (cf. O'Malley and Jameson (1975)) also show that the limiting control has a negative delta function impulse at $t = 0$. We note that the limiting solution for $t > 0$ wouldn't be trivial if $x_1(0) \neq 0$ and that introduction of the ϵ provides a more convenient method of finding the limiting singular arc solution than the more familiar technique of differentiating the optimality condition $H_u = 0$ twice with respect to t (cf., e.g., Robbins (1967)).

The preceding solution wouldn't be appropriate for a finite control bound m, because the impulse in the initial boundary layer would exceed this bound Instead, standard maximum principle arguments (cf. Boltyanskii (1971)) show that the continuous optimal control is determined by

$$\begin{cases} u = -p_2/\epsilon^2 \quad \text{if} \quad |p_2| < m\epsilon^2 \\ \text{and} \\ |u| = m \quad \text{otherwise.} \end{cases}$$

Moreover, the state and costate vectors satisfy the canonical equations (2.3) as before. Anticipating that the control initially saturates at its negative bound, we might seek a solution

$$u = \begin{cases} -m, & 0 \leq t \leq t_1 \\ -p_2/\epsilon^2, & t_1 \leq t \leq 5 \end{cases} \tag{2.5}$$

and determine t_1, if possible, so that saturation does not occur on $(t_1,5)$ while $|p_2| > m\epsilon^2$ on $(0,t_1)$. Clearly,

$$x_1(t) = -\frac{m}{2}t^2 + t, \quad x_2(t) = -mt + 1 \tag{2.6}$$

for $0 \leq t < t_1$ while we must satisfy the singularly perturbed problem

$$\begin{cases} \dot{x}_1 = x_2, \quad x_1(t_1) = x_1(t_1^-) \\ \epsilon^2\dot{x}_2 = -p_2, \quad x_2(t_1) = x_2(t_1^-) \\ \dot{p}_1 = -x_1, \quad p_1(5) = 0 \\ \dot{p}_2 = -x_2 - p_1, \quad p_2(t_1) = m\epsilon^2, \quad p_2(5) = 0 \end{cases} \tag{2.7}$$

for $t_1 \leq t \leq 5$.

The system (2.7) has eigenvalues $\pm 1/k(\epsilon)$, $\pm k(\epsilon)/\epsilon$, where $k^2(\epsilon) = (1 + \sqrt{1-4\epsilon^2})/2$. Thus, the solution of the two-point problem is of the form

$$(2.8)\qquad \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -1/k \\ -1/k^2 \\ 1 \\ \epsilon^2/k^3 \end{pmatrix} Ae^{(t-t_1)/k} + \begin{pmatrix} 1/k \\ -1/k^2 \\ 1 \\ -\epsilon^2/k^3 \end{pmatrix} Be^{-(t-t_1)/k}$$

$$+ \epsilon \begin{pmatrix} -\epsilon/k^2 \\ 1/k \\ -\epsilon^2/k^3 \\ \epsilon \end{pmatrix} Ce^{-k(t-t_1)/\epsilon} + \epsilon \begin{pmatrix} -\epsilon/k^2 \\ -1/k \\ \epsilon^2/k^3 \\ \epsilon \end{pmatrix} De^{-k(5-t)/\epsilon}.$$

The five boundary conditions in (2.7) provide four linear equations for A, B, C, and D and a nonlinear equation for t_1. For $5 - t_1 >> \epsilon$, the exponential $e^{-k(5-t_1)/\epsilon}$ is exponentially negligible, so we successively find

$$A \approx -Be^{-2(5-t_1)}, \qquad D \approx 2Be^{-(5-t_1)},$$

$$C \approx m + B(1 + e^{-2(5-t_1)}), \text{ and } B \approx \tfrac{1}{2}(x_1(t_1) - x_2(t_1))$$

while t_1 must satisfy

$$e^{-2(5-t_1)} \approx \frac{mt_1^2+2(m-1)t_1-2}{mt_1^2-2(m+1)t_1+2}.$$

It follows that there is a unique root t_1 such that $0 < t_1 < \frac{1}{m} - 1 + \sqrt{1 + \frac{1}{m^2}}$.

Indeed for $m = 1$, t_1 is very near the upper bound $\sqrt{2}$, so the optimal control is approximately given by

$$(2.9) \qquad u \approx \begin{cases} -1, & 0 \leq t < \sqrt{2} \\ (\sqrt{2} - 1)e^{-(t-\sqrt{2})}, & t > \sqrt{2}. \end{cases}$$

We solved problem (2.1) with $\epsilon = 0$ by our asymptotic technique, which is discussed in Section 3, and found that our results (see Figure 2) qualitatively agreed with the computed solution pictured in Jacobson et al. (1970). We found a minimum cost of 0.379 compared to their value of 0.414. Anderson (1972) solved this problem by a search technique, obtaining a switching time of 1.414 with corresponding trajectory values of 0.4144 and -0.4136, compared to our values of 1.414, 0.4144, and -0.4138, respectively.

3. A SYSTEMATIC APPROACH TO CASE ONE PROBLEMS

Let us now consider free endpoint problems where

$$(3.1) \qquad b'Qb > 0$$

for a prescribed initial state vector $x(0)$. Since the vector b has rank one, we can transform it to its row echelon form

$$\tilde{b} = Mb = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

by a nonsingular matrix M. (Here, the zero is an $(n - 1)$-vector.) Setting

$$\tilde{x} = Mx$$

we get a transformed problem analogous to (1.1)-(1.2) with

$$\tilde{A} = MAM^{-1}, \quad \tilde{Q} = (M')^{-1}QM^{-1} > 0,$$

$$\tilde{b}'\tilde{Q}\tilde{b} = b'QB > 0, \quad \text{and} \quad \tilde{u} = u.$$

Let us assume now that such transformations have already occurred so that

$$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.2}$$

and, using corresponding partitioning, write

$$A = \begin{bmatrix} A_{11} & a_{12} \\ a'_{21} & a_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & q_{12} \\ q'_{12} & q_{22} \end{bmatrix}, \tag{3.3}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{and} \quad p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

where a_{12}, a_{21}, q_{12}, x_1 and p_1 are $(n - 1)$-vectors and a_{22}, q_{22}, x_2, and p_2 are scalars with $q_{22} > 0$. The canonical equations then take the form

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 = a'_{21}x_1 + a_{22}x_2 + u \\ \dot{p}_1 = -Q_{11}x_1 - q_{12}x_2 - A'_{11}p_1 - a_{21}p_2 \\ \text{and} \\ \dot{p}_2 = -q'_{12}x_1 - q_{22}x_2 - a'_{12}p_1 - a_{22}p_2 \end{cases} \tag{3.4}$$

subject to the end conditions that $x(0)$ is prescribed and $p(T) = 0$.

With the control constraint $|u| \leq m$, the maximum principle implies that the control is either saturated or $u = -p_2/\epsilon^2$. Since the unconstrained problem has an optimal control which is initially unbounded and then follows a singular arc as $\epsilon \to 0$, it is natural to seek a bounded control which is saturated on an initial interval $0 \leq t \leq t_1$ and unsaturated for $t_1 < t \leq T$. For such switching solutions, we must satisfy the initial value problem

$$(3.5) \qquad \begin{cases} \dot{x}_1 = A_{11}x_1 + a_{12}x_2, & x_1(0) \text{ given} \\ \\ \dot{x}_2 = a_{21}'x_1 + a_{22}x_2 \pm m, & x_2(0) \text{ given} \end{cases}$$

for $0 \leq t \leq t_1$ and the singularly perturbed two-point problem

$$(3.6) \qquad \begin{cases} \dot{x}_1 = A_{11}x_1 + a_{12}x_2, \quad x_1(t_1^+) = x_1(t_1^-) \\ \epsilon^2\dot{x}_2 = \epsilon^2 a_{21}'x_1 + \epsilon^2 a_{22}x_2 - p_2, \quad x_2(t_1^+) = x_2(t_1^-) \\ \dot{p}_1 = -Q_{11}x_1 - q_{12}x_2 - A_{11}'p_1 - a_{21}p_2, \quad p_1(T) = 0 \\ \dot{p}_2 = -q_{12}'x_2 - q_{22}x_2 - a_{12}'p_1 - a_{22}p_2, \quad p_2(t_1) = \mp m\epsilon^2, \qquad p_2(T) = 0 \end{cases}$$

for $t_1 \leq t \leq T$ where the switching time t_1 is still unspecified. Besides requiring continuity of the states, costates, and control at t_1, we must check tha the control remains saturated until t_1 and unsaturated thereafter. One would expect that the sign of the initial impulse for the unconstrained problem would generally predict which control bound would initially saturate.

Our previous experience with singular perturbation problems shows that the asymptotic solution on $t_1 \leq t \leq T$ will be of the form

$$(3.7)\qquad \begin{cases} x_1(t,\epsilon) = X_1(t,\epsilon) + \epsilon^2 m_1(\tau,\epsilon) + \epsilon^2 n_1(\sigma,\epsilon) \\ x_2(t,\epsilon) = X_2(t,\epsilon) + \epsilon m_2(\tau,\epsilon) + \epsilon n_2(\sigma,\epsilon) \\ p_1(t,\epsilon) = P_1(t,\epsilon) + \epsilon^2 f_1(\tau,\epsilon) + \epsilon^2 g_1(\sigma,\epsilon) \\ p_2(t,\epsilon) = \epsilon^2 P_2(t,\epsilon) + \epsilon^2 f_2(\tau,\epsilon) + \epsilon^2 g_2(\sigma,\epsilon) \end{cases}$$

where the functions of

$$\tau = (t - t_1)/\epsilon \quad \text{or} \quad \sigma = (T - t)/\epsilon$$

tend to zero as that "stretched variable" tends to infinity (cf. O'Malley and Jameson (1975), noting that we need $p_2 = O(\epsilon^2)$). Within (t_1,T), the solution will be asymptotically represented by the outer solution

$$(3.8)\qquad (X_1, X_2, P_1, \epsilon^2 P_2)$$

which has a power series expansion in ϵ. Its leading term $(X_{10}, X_{20}, P_{10}, \epsilon^2 P_{20})$ will lie along a singular arc of order one. We note that the initial boundary layer term $f_2(\tau,0)$ allows the nonuniform convergence of the control $u = -p_2/\epsilon^2$ as $\epsilon \to 0$ at the switching point t_1 from $+m$ to $-P_{20}(t_1^+)$, i.e., the jump from its constrained value to its limiting value along the singular arc. Analogously, the terminal boundary layer term $g_2(\sigma,0)$ allows a jump in p_2/ϵ^2 from the singular arc value $P_{20}(T^-)$ to the terminal value zero. Explicit calculation of the boundary layer terms will not be required.

Since (3.8) must satisfy (3.6), the limiting outer solution will necessarily satisfy the reduced problem

$$
(3.9)\qquad \begin{cases} \dot{X}_{10} = A_{11}X_{10} + a_{12}X_{20}, & X_{10}(t_1) = x_1(t_1^-) \\ \dot{X}_{20} = a_{21}'X_{10} + a_{22}X_{20} - P_{20}, & X_{20}(t_1) = x_2(t_1^-) \\ \dot{P}_{10} = -Q_{11}X_{10} - q_{12}X_{20} - A_{11}'P_{10}, & P_{10}(T) = 0 \\ 0 = -q_{12}'X_{10} - q_{22}X_{20} - a_{12}'P_{10} & \end{cases}
$$

for $t_1 < t < T$. We can solve immediately for X_{20} and P_{20} as functions of X_{10} and P_{10} (since $q_{22} \neq 0$). This leaves us with the linear two-point problem

$$
(3.10)\qquad \begin{cases} \dot{X}_{10} = AX_{10} - SP_{10}, & X_{10}(t_1) = x_1(t_1^-) \\ \dot{P}_{10} = -QX_{10} - A'P_{10}, & P_{10}(T) = 0 \end{cases}
$$

for $A = A_{11} - a_{12}q_{22}^{-1}q_{12}'$, $Q = Q_{11} - q_{12}q_{22}^{-1}q_{12}' \geq 0$, and $S = a_{12}q_{22}^{-1}a_{12}' \geq 0$. As usual, it is convenient to solve this standard regulator problem by setting

$$
(3.11)\qquad P_{10}(t) = K(t)X_{10}(t)
$$

where the $(n-1) \times (n-1)$ matrix K satisfies the Riccati differential equation

$$
(3.12)\qquad \dot{K} + KA + A'K + Q = KSK, \quad K(T) = 0.
$$

Standard arguments (cf., e.g., Athans and Falb) guarantee the existence of a unique symmetric solution $K \geq 0$ for all $t < T$. Thus, X_{10} satisfies the initial value problem

$$
(3.13)\qquad \dot{X}_{10} = (A - SK)X_{10}, \quad X_{10}(t_1) = x_1(t_1^-)
$$

while

(3.14) $$X_{20} = -q_{22}^{-1}(q_{12}' + a_{12}'K)X_{10}, \quad X_{20}(t_1) = x_2(t_1^-).$$

The initial values $x_1(t_1^-)$ and $x_2(t_1^-)$ must be determined by integrating the saturated control problem (3.5) on $[0,t_1]$ and determining the switching time t_1 by solving the nonlinear scalar equation

(3.15) $$x_2(t_1^-) = -q_{22}^{-1}(q_{12}' + a_{12}'K(t_1))x(t_1^-).$$

For certain simple problems (like our preceding examples), t_1 can be explicitly determined. In general, however, one must attempt to solve (3.15) for its least positive root t_1 numerically.

To check that our solution candidate remains saturated on $0 \leq t \leq t_1$, we integrate the linear system

(3.16) $$\begin{cases} \dot{p}_1 = -A_{11}'p_1 - a_{12}p_2 - Q_{11}x_1 - q_{12}x_2, & p_1(t_1^-) = P_{10}(t_1^+) = K(t_1)x_1(t_1^-) \\ \dot{p}_2 = -a_{12}'p_1 - a_{22}p_2 - q_{12}'x_1 - q_{22}x_2, & p_2(t_1^-) = 0 \end{cases}$$

backwards from t_1, insisting that $p_2 \neq 0$ on $0 \leq t < t_1$ (the nonhomogeneous terms are known, since x_1 and x_2 follow from (3.5)). Likewise, we must be sure that p_2 remains unsaturated for $t > t_1$, i.e.,

(3.17) $$|P_{20}| = |a_{21}'X_{10} + a_{22}X_{20} - \dot{X}_{20}| < m$$

for $t_1 < t < T$.

Our procedure then shows us how to produce candidates for the optimal solution of Case 1 problems. Examples can surely be found (cf. next section) where the troublesome possibilities mentioned above eliminate some potential solution candidates. For other problems, it produces suboptimal controls which may still be of value. Occasionally, we can be sure that a computed solution is optimal. An example, on an infinite time interval, is

$$\begin{cases} \dot{x}_1 = x_2 + u, \quad x_1(0) = 0 \\ \dot{x}_2 = -u, \quad x_2(0) = -0.5, \quad |u| \le 1 \\ J = \frac{1}{2}\int_0^T x_1^2 dt \quad \text{with} \quad x_1(T) = x_2(T) = 0 \quad \text{for } T \text{ free} \end{cases}$$

(*cf.* Johnson and Gibson (1963)). We should note how effective our method is in selecting the singular arc solution and that it relates to the popular technique of synthesizing a control by integrating backwards from terminal time.

Algorithm

We have used the above asymptotic analysis to construct the following algorithm to find numerical solutions of Case 1 problems.

1. Solve the Riccati differential equation (3.12) backwards from $t = T$ to $t = 0$. [We used Gear's code (1971) for all of our numerical integrations; however, any other good code would have sufficed.]
2. Decide which way the control bound will initially saturate and select an initial guess for the switching time t_1.
3. Integrate equations (3.5) from $t = 0$ to the latest guess for the switching time. We repeat this step until equation (3.15) is satisfied to a sufficient degree of accuracy. We generate successive guesses for t_1 using a Newton-like procedure due to Brown (1969). If a negative root is found, it can be eliminated by deflation so that a second iteration can be attempted. Also, if the iteration is diverging the sign of the control bound can be reversed and the procedure repeated.

4. Once the switching time has been found, we integrate equations (3.16) backwards from t_1 to determine the costate vector in the saturated region $[0,t_1]$. We then integrate equation (3.13) forward from t_1 and use (3.14), (3.11), and the second of (3.9) to determine the solution in the unsaturated region. During the integration, we check that the appropriate bounds on p_2 are not exceeded.
5. Finally, calculate the cost by integrating (1.2) using the trapezoidal rule.

4. TWO MORE DIFFICULT PROBLEMS

a. Sirisena (1970) considered the Case 1 problem

$$\text{(4.1)} \qquad \begin{cases} \dot{x}_1 = x_2, \quad x_1(0) = 2 \\ \dot{x}_2 = u, \quad x_2(0) = 0 \\ J = \frac{1}{2}\int_0^T [(x_1 + x_2)^2 + \epsilon^2 u^2]dt, \quad |u| \leq 1 \end{cases}$$

for $\epsilon = 0$. If we, instead, determine the asymptotic solution as $\epsilon \to 0$ as in Section 3, we determine the switching time

$$t_1 = \sqrt{5} - 1 \approx 1.236 \quad (\text{for } T >> t_1)$$

where the control changes from $u = -1$ to the singular arc. Specifically, the limiting control is

$$u = t_1 e^{-(t-t_1)} \quad \text{for} \quad t_1 < t < T$$

and the corresponding limiting cost is

$$J = \frac{1}{15}(25\sqrt{5} - 41) \approx 0.993447.$$

This control would seem to compare favorably to Sirisena's bang-bang solution for the fixed endpoint problem with $x(T) = 0$, achieved with an infinite number of switchings with first switching point 1.227 and cost 0.993455. If we take our constraint seriously, however, we must rule out this control since $u(t_1^+) = \sqrt{5} - 1$ exceeds the bound $u = 1$, (see Figure 3).

Our nearly good solution suggests that we might instead seek a solution where the control switches from $u = -1$ on some interval $0 \leq t < t_1$ to $u = 1$ on $t_1 < t < t_2$ after which u follows a singular arc. An unsaturated control must be constructed to smoothly switch from -1 to 1 near t_1 (i.e., from $p_2 = \epsilon^2$ to $-\epsilon^2$). This type of interior (or transition) layer differs somewhat from our earlier boundary layers because the desired limits occur for finite negative and positive values, $-\eta_-$ and η_+, of the appropriate stretched variable $\eta = (t - t_1)/\epsilon^2$, not for $\eta = \pm\infty$. Moreover, the limiting solution is unrelated to a singular arc. For these reasons, the local behavior is determined by a regular perturbation procedure on $(-\eta_-,\eta_+)$, although the solution will still exhibit singular perturbation features (i.e., nonuniform convergence) because the t interval corresponding to $(-\eta_-,\eta_+)$ vanishes as $\epsilon \to 0$. As before, the boundary layer at t_2 is studied by a singular perturbation analysis which allows the control to switch from its constraint set to a singular arc.

For (4.1), the canonical equations are

$$(4.2)\qquad \begin{cases} \dot{x}_1 = x_2, & \dot{x}_2 = u \\ \dot{p}_1 = -x_1 - x_2, & \dot{p}_2 = -x_1 - x_2 - p_1. \end{cases}$$

On $0 \leq t < t_1^-$, then, the solution is

$$(x_1,x_2,u) = (2 - \tfrac{1}{2}t^2, -t, -1).$$

For $-\epsilon^2\eta_- \le t - t_1 \le \epsilon^2\eta_+$, we have $u = -p_2/\epsilon^2$ and $p_2(t_1 \mp \epsilon^2\eta_\mp) = \pm\epsilon^2$, $p_2(t_1) = 0$. Moreover, we'd locally obtain x_1, x_2, p_1, and p_2 as power series in ϵ with x_1, x_2, and p_1 having nearly the constant values $x_1(t_1^-)$, $x_2(t_1^-)$, and $p_2(t_1^-)$. Since the limiting value of p_2 jumps symmetrically from $-\eta_-$ to η_+, we have $\eta_- = \eta_+ = (|x_1(t_1^-) + x_2(t_1^-) + p_1(t_1^-)|)^{-1}$, provided the denominator is nonzero. Proceding to $t_1^+ < t < t_2$, we must have the limiting solution

$$(x_1,x_2,u) = (x_1(t_1^-) + x_2(t_1^-)(t - t_1^+) + (t - t_1^+)^2/2,\ x_2(t_1^-) + t - t_1^+, 1)$$

and $p_2(t_2^-) = \epsilon^2 = p_2(t_1^+)$. For $t > t_2$, we'll have $u = -p_2/\epsilon^2$ and (4.2) implies that the limiting solution $(X_{10}, X_{20}, P_{10}, \epsilon^2 P_{20})$ must satisfy the limiting singular arc problem

$$\begin{cases} \dot{X}_{10} = X_{20}, & X_{10}(t_2) = x_1(t_2^-) \\ \dot{X}_{20} = -P_{20}, & X_{20}(t_2) = x_2(t_2^-) \\ \dot{P}_{10} = -X_{10} - X_{20}, & P_{10}(T) = 0 \\ 0 = -X_{10} - X_{20} - P_{10}. & \end{cases}$$

Thus, $X_{20} = -X_{10} - P_{10}$, $P_{20} = -\dot{X}_{20} = -\ddot{X}_{10}$, and there remains the two point problem $\dot{X}_{10} = -X_{10} - P_{10}$, $\dot{P}_{10} = P_{10}$, $X_{10}(t_2) = x_1(t_2^-)$, $P_{10}(T) = 0$. Hence, we follow the singular arc

$$
(4.3) \qquad \begin{cases} X_{10}(t) = x_1(t_2^-)e^{-(t-t_2)} \\ \\ P_{10}(t) = 0, \quad X_{20} = P_{20} = -X_{10} \end{cases}
$$

for $t > t_2$. In order for the control to remain unsaturated there, we must have

$$
|x_1(t_2^-)| < 1.
$$

Lastly, we obtain p_1 and p_2 for $t < t_2$ by integrating the costate equations of (4.2) backwards with the limiting boundary values $p_1(t_2^-) = 0$, $p_2(t_2^-) = 0$. Since x_1 and x_2 are defined differently for $0 \leq t < t_1$ and $t_1 < t < t_2$, we also have different expressions for p_1 and p_2 on these intervals. Our solution is now completely determined up to specification of the switching times t_1 and t_2, $0 < t_1 < t_2$.

The switching times follow from the two scalar equations

$$
(4.4) \qquad \begin{cases} x_2(t_2^-) = x_2(t_2^+) \\ \text{and} \\ p_2(t_1) = 0 \end{cases}
$$

which reduce to

$$
\frac{1}{2}(\Delta t)^2 + (1 - t_1)\Delta t + (2 - t_1 - \frac{1}{2}t_1^2) = 0
$$

and

$$
\frac{1}{8}(\Delta t)^3 + \frac{1}{6}(3 - 2t_1)(\Delta t)^2 + \frac{1}{2}(3 - 2t_1 - \frac{1}{2}t_1^2)\Delta t + (2 - t_1 - \frac{1}{2}t_1^2) = 0
$$

for positive $\Delta t = t_2 - t_1$. Solving these equations simultaneously we obtain $t_1 = 1.227$ and $t_2 = 1.564$ and a resulting cost 0.993455. We further check that p_2 has no root other than t_1 on $0 < t < t_2$ and that $|p_2(t_2^+)| = |x_1(t_2^-)| < 1$. We note that these results (shown in Figure 4) are in agreement with those of Sirisena (1970), who needed an infinite number of switchings. Presumably, an analogous procedure could be used for other Case 1 problems requiring a finite number of switchings.

b. Jacobson et al. (1970) considered the Case 2 problem

$$(4.5)\qquad \begin{cases} \dot{x}_1 = x_2, & x_1(0) = 0 \\ \dot{x}_2 = u, & x_2(0) = 1 \\ J(\epsilon) = \frac{1}{2}\int_0^5 (x_1^2 + \epsilon^2 u^2)dt, & |u| \le 1 \end{cases}$$

with $\epsilon = 0$. Since the simplest Case 1 solutions involve a control which switches from being saturated to lying on a singular arc, we *might attempt* to find Case 2 solutions which switch once between the two control bounds and then follow the singular arc. (For this problem, it will be impossible to find a control which switches directly from its bound to the singular arc.) Noting that the unconstrained problem has a large negative initial control as $\epsilon \to 0$ (cf. O'Malley and Jameson (1976)), we seek a solution such that $u = -1$ for $0 \le t < t_1$, $u = 1$ for $t_1 < t < t_2$, and $u = -p_2/\epsilon^2$ for t near t_1 and for $t_2 < t < 5$. On $(t_2,5)$, the limiting solution $(x_{10}, x_{20}, p_{10}, p_{20})$ will satisfy the limiting canonical system

$$(4.6)\qquad \dot{x}_{10} = x_{20},\quad 0 = -p_{20},\quad \dot{p}_{10} = -x_{10}, \text{ and } \dot{p}_{20} = -p_{10},$$

i.e., it will follow the trivial second order singular arc. Integrating the state equations (4.5), we obtain

$$(x_1,x_2) = (t - \tfrac{1}{2}t^2, 1 - t) \qquad \text{for} \quad 0 < t < t_1$$

and

$$(x_1,x_2) = (x_1(t_1^+) + x_2(t_1^+)(t-t_1) + \tfrac{1}{2}(t-t_1)^2, \;\; x_2(t_1^+) + (t-t_1)) \quad \text{for} \quad t_1 < t < t_2.$$

As for the preceding problem the transition layer at t_1 and the boundary layer at t_2 will allow discontinuities in u, but not in the states x_1 and x_2. Thus we must have $x_i(t_1^-) = x_i(t_1^+)$ and $x_i(t_2^-) = x_{i0}(t_2^+) = 0$, $i = 1, 2$. These conditions imply that

$$t_1 = 1 + \frac{\sqrt{2}}{2} \quad \text{and} \quad t_2 = 1 + \sqrt{2}. \tag{4.7}$$

The resulting solution (see Figure 5) compares very favorably to that of Jacobson et al; the cost being 0.269 compared to their value 0.277.

Regrettably, however, the control just computed is suboptimal. It does remain unsaturated on $(t_2,5)$, but the costate vector must satisfy the limiting terminal value problem

$$\begin{cases} \dot{p}_1 = -x_1, & p_1(t_2^-) = 0 \\ \dot{p}_2 = -p_1, & p_2(t_2^-) = 0 \end{cases} \tag{4.8}$$

for $t < t_2$. Since $x_1 > 0$ within $(0,t_2)$, p_1 and p_2 are also positive there, so we can't achieve the optimality condition $p_2(t_1) = 0$. (Our algorithm yields $p_2(t_1^+) = 0.0104$). This closure problem occurs because our switching requirements completely determine the state, costate, and switching times. The additional optimality condition is an extra constraint without a corresponding degree of freedom.

It should be realized that the optimal control for this problem presumably involves an infinite number of switchings which accumulate at a point after which the control follows a singular arc (cf. Fuller (1963), Grensted and Fuller (1965), and Marchal (1973)), although fewer switchings may be possible for special initial values. Our suboptimal results seem very good, however. They relate to Sirisena's concept of an optimal r-switch solution (cf. Sirisena (1970)).

A related Case 2 example with a trivial singular arc requiring an infinite number of switchings is

$$\begin{cases} \dot{x}_1 = x_2, & x_1(0) = 2 \\ \dot{x}_2 = u, & x_2(0) = -2 \\ J = \frac{1}{2}\int_0^4 x_1^2 dt, & |u| \leq 1 \end{cases}$$

(cf. Marchal (1973)). When we try to calculate a one switch solution starting at $u = -1$, we get $t_1 = 2$, $\Delta t = t_2 - t_1 = 0$, and cost 1.6. The optimal solution has $u = 1$ until $t = 0.057$, switches to $u = -1$ until about $t = 2.6$, and then has an infinite number of switchings accumulating at $t = 3.43$ after which $u = 0$. The optimal cost is 1.52. Our suboptimal solution is presumably not as good as for the preceding problem where we were able to switch once before meeting the singular arc.

5. COMMENTS ON HIGHER ORDER SINGULAR ARCS

Experts (cf. e.g., Krener (1976)) feel that infinite switching is generic for problems with singular arcs of order p, $p \geq 2$, although finite switching may be possible for special initial conditions. Our preceding discussion supports this opinion. For such problems, we can readily identify the appropriate form of singular arc solution through our singular perturbation analysis, and can generally find a suboptimal $p - 1$ switch solution converging to the singular arc. Suboptimal solutions with more switchings (a la Sirisena) might also be sought.

Specifically, for Case 2, we have

$$b'Qb = 0 \quad \text{and} \quad b'A'QAb > 0.$$

The problem can be transformed so that

$$(5.1) \qquad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & a_{12} & 0 \\ a_{21}' & a_{22} & 1 \\ a_{31}' & a_{32} & a_{33} \end{pmatrix}, \quad \text{and} \quad Q = \begin{pmatrix} Q_{11} & q_{12} & 0 \\ q_{12}' & q_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where A_{11} and Q_{11} are matrices of dimension $(n - 2) \times (n - 2)$ and q_{22} is a positive scalar (cf. O'Malley and Jameson (1976)). Introducing corresponding state and costate vectors

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix},$$

the scalar optimal control either lies on its bound $|u| = m$ or else $u = -p_3/\epsilon^2$. On the singular arc, the control should be unsaturated and the canonical system has the form

$$(5.2)\qquad \begin{cases} \dot{x}_1 = A_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 = a'_{21}x_1 + a_{22}x_2 + x_3 \\ \epsilon^2\dot{x}_3 = \epsilon^2(a'_{31}x_1 + a_{32}x_2 + a_{33}x_3) - p_3 \\ \dot{p}_1 = -Q_{11}x_1 - q_{12}x_2 - A'_{11}p_1 - a_{21}p_2 - a_{31}p_3 \\ \dot{p}_2 = -q'_{12}x_1 - q_{22}x_2 - a'_{12}p_1 - a_{22}p_2 - a_{32}p_3 \\ \dot{p}_3 = -p_2 - a_{33}p_3. \end{cases}$$

Seeking a singular arc solution of the form

$$(5.3)\qquad (x_1,x_2,x_3,p_1,p_2,p_3) = (X_1,X_2,X_3,P_1,\epsilon^2P_2,\epsilon^2P_3)$$

where the X_i's and P_i's have asymptotic series expansions in $\sqrt{\epsilon}$, the leading terms will satisfy

$$\begin{cases} \dot{X}_{10} = A_{11}X_{10} + a_{12}X_{20} \\ \dot{X}_{20} = a'_{21}X_{10} + a_{22}X_{20} + X_{30} \\ \dot{X}_{30} = a'_{31}X_{10} + a_{32}X_{20} + a_{33}X_{30} - P_{30} \\ \dot{P}_{10} = -Q_{11}X_{10} - q_{12}X_{20} - A'_{11}P_{10} \\ 0 = -q'_{12}X_{10} - q_{22}X_{20} - a'_{12}P_{10} \\ \dot{P}_{30} = -P_{20} - a_{33}P_{30} \end{cases}$$

so we must have

$$(5.4)\quad \begin{cases} X_{20} = -q_{22}^{-1}q_{12}'X_{10} - q_{22}^{-1}a_{12}'P_{10} \\ X_{30} = \dot{X}_{20} - a_{21}'X_{10} - a_{22}X_{20} \\ P_{30} = a_{31}'X_{10} + a_{32}X_{20} + a_{33}X_{30} - \dot{X}_{30} \\ \text{and} \\ P_{20} = -a_{33}P_{30} - \dot{P}_{30} \end{cases}$$

and there remains the linear $2(n-2)$-dimensional problem

$$(5.5)\quad \begin{cases} \dot{X}_{10} = (A_{11} - a_{12}q_{22}^{-1}q_{12}')X_{10} - a_{12}q_{22}^{-1}a_{12}'P_{10} \\ \dot{P}_{10} = -(Q_{11} - q_{12}q_{22}^{-1}q_{12}')X_{10} - (A_{11}' - q_{12}q_{22}^{-1}a_{12}')P_{10} \end{cases}$$

analogous to (3.10). Since $Q \geq 0$ and $q_{22} > 0$ imply that $Q_{11} - q_{12}q_{22}^{-1}q_{12}' \geq 0$, standard results imply that the two point problem with $X_{10}(\tilde{t})$ prescribed and $P_{10}(T) = 0$ will have a unique solution on any interval $[\tilde{t},T]$. Thus, the singular arc solution with control $u \approx -P_{30}$ can be easily computed on any such interval provided the state $x_1(\tilde{t}^-)$ is known from the initial trajectory $[0,\tilde{t})$ with bang-bang control. In particular, Case 2 problems converge to the trivial singular arc solution when $n = 2$. The Case 2 problem

$$\begin{cases} \dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u, \quad x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = 0 \\ J = \frac{1}{2}\int_0^5 (x_1 + x_2)^2 dt, \quad |u| \leq 1 \end{cases}$$

will, for example, have a nontrivial singular arc (cf. Sirisena (1970)).

We can analogously determine the singular arcs appropriate for Case p problems, $p > 2$. For example, we find the trivial singular arc for the Case 3 problem

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u, \quad x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = 0$$

$$J = \frac{1}{2}\int_0^5 x_1^2 dt, \quad |u| \leq 1$$

(cf. Grensted and Fuller (1965) and Sirisena (1974)). The optimal solution switches infinitely often before landing on the singular arc and has an optimum cost 1.2521. Our simplest suboptimal solution switches at $t_1 \approx .794$ and $3t_1$ and reaches the singular arc at $4t_1$ with a cost of 1.2665.

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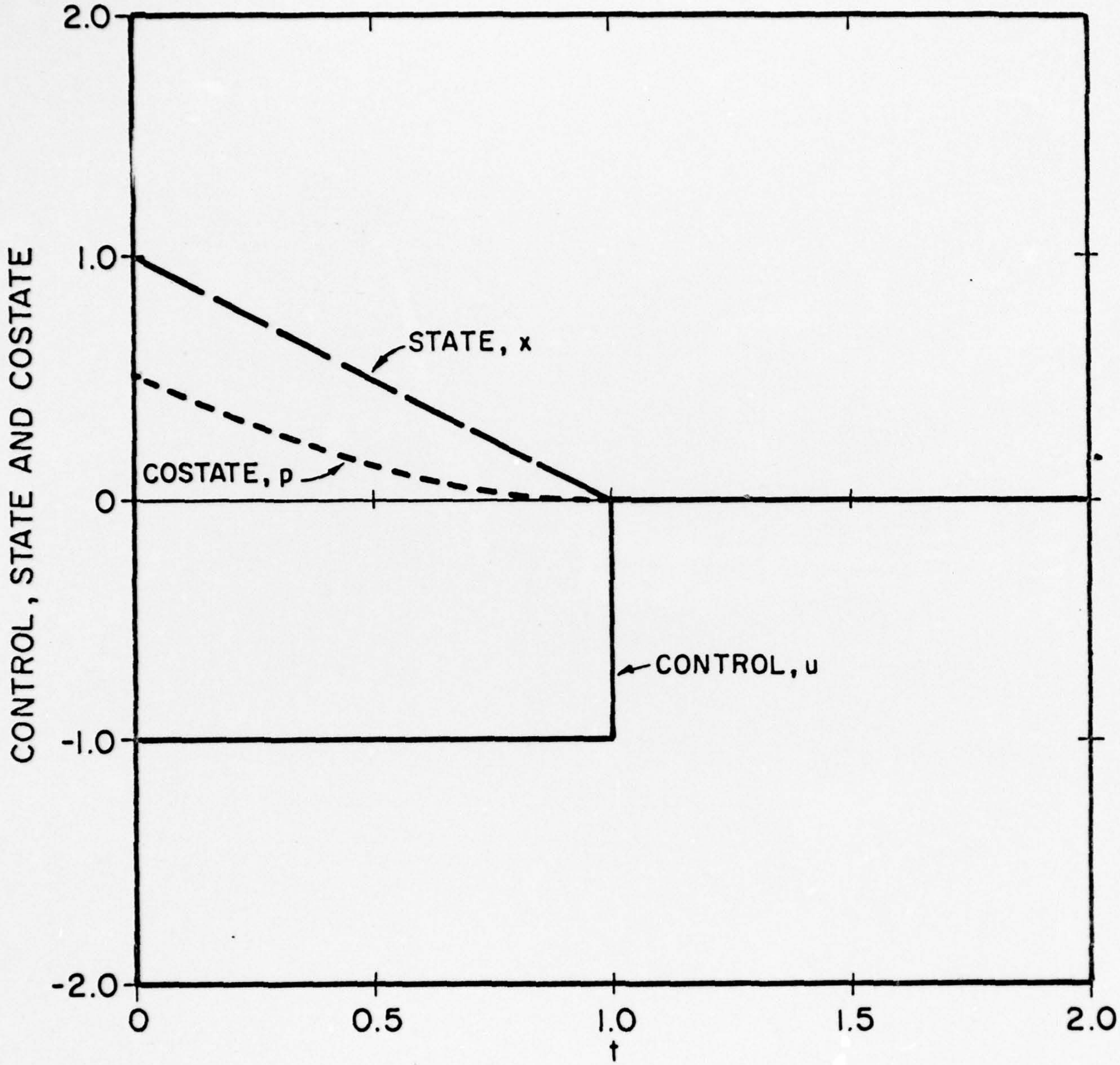

FIGURE 1: Example 2.a

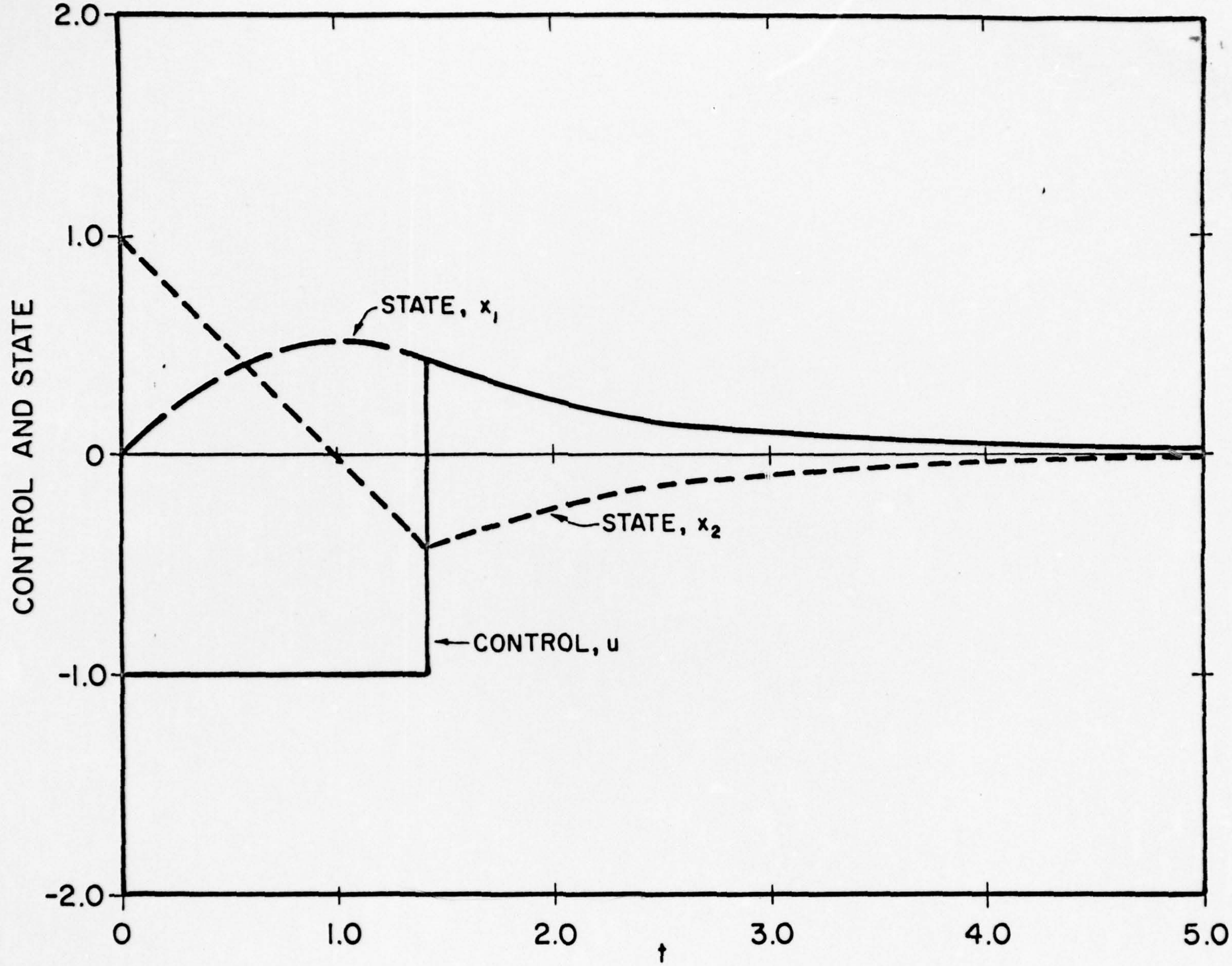

FIGURE 2: Example 2.b

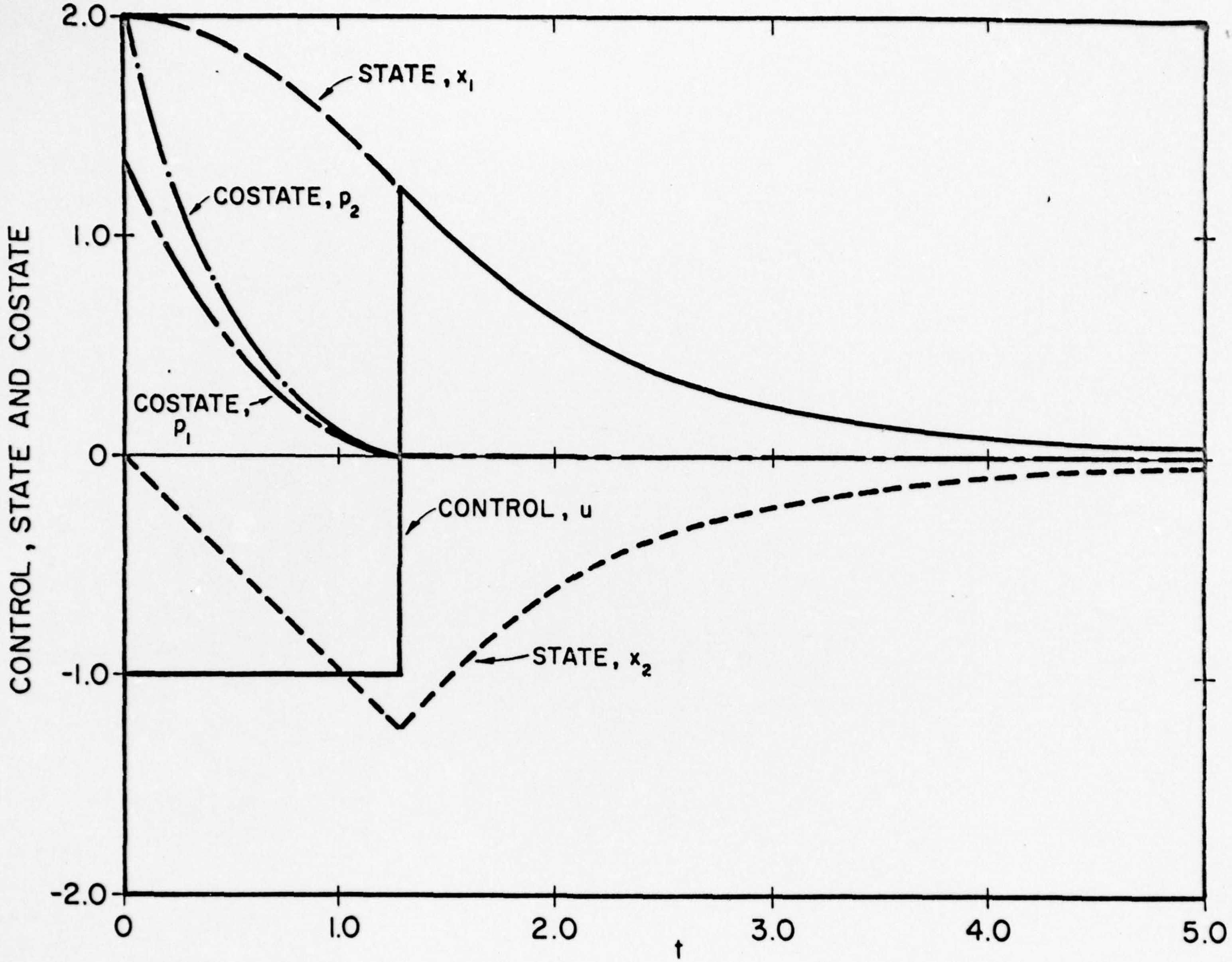

FIGURE 3: Example 4.a with one switch

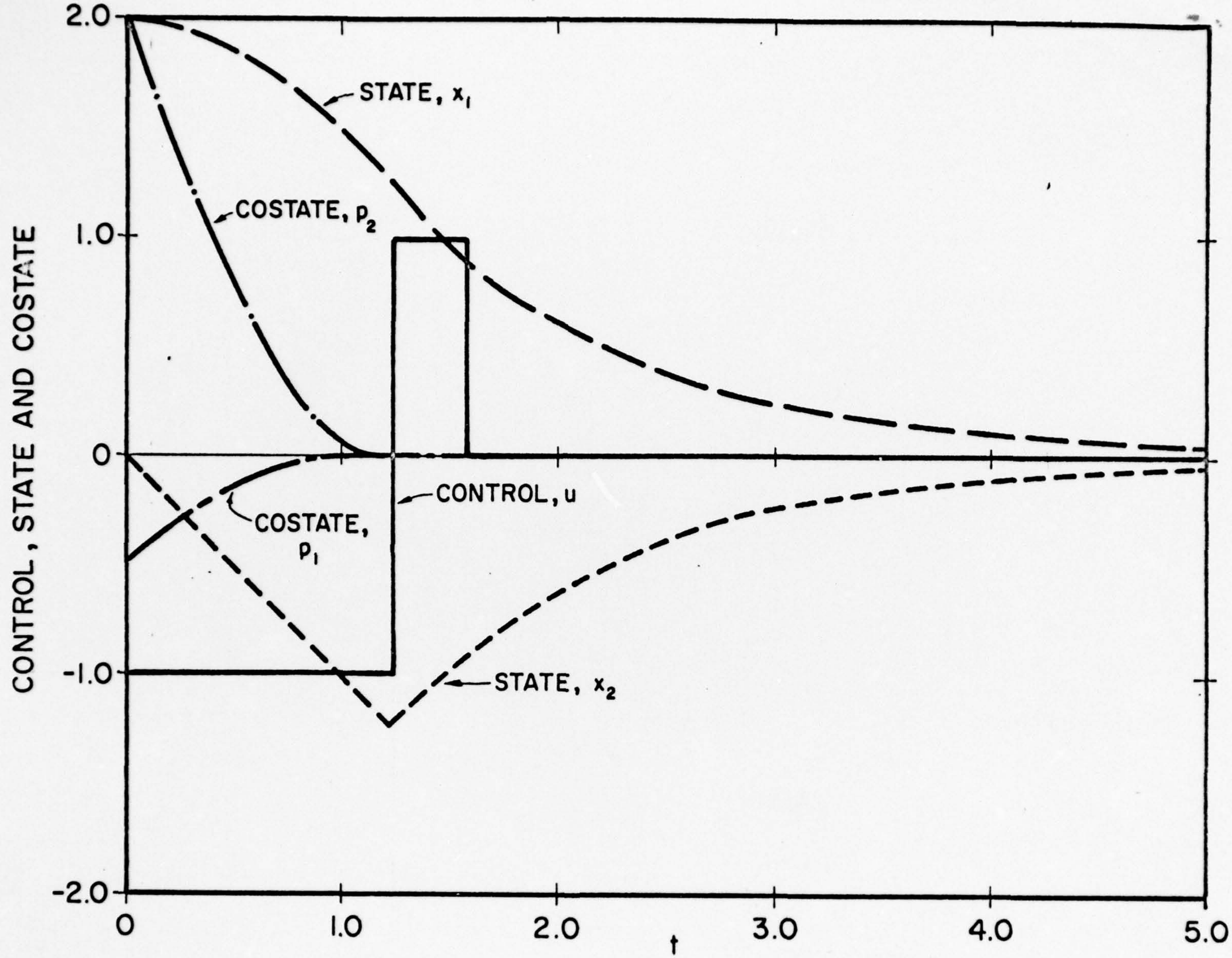

FIGURE 4: Example 4.a with two switches

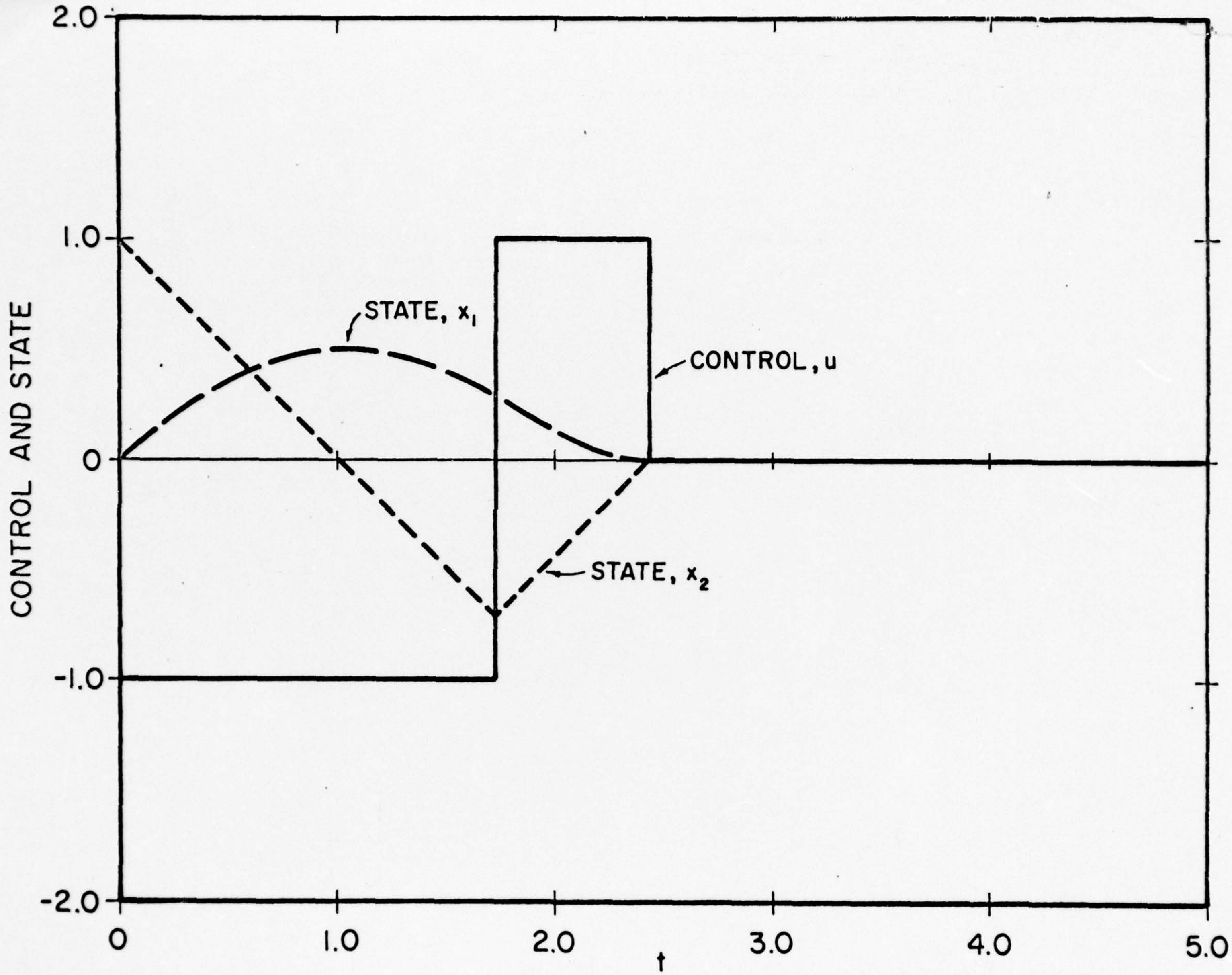

FIGURE 5: Example 4.b

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We consider singular optimal control problems consisting of a state equation

x dot

$$\dot{x} = Ax + Bu$$

for vectors x and scalars u and a cost functional → next page

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Block 20/Abstract

$$J = \frac{1}{2} \int_0^T (x'Qx + \varepsilon^2 u^2)dt$$

to be minimized, for $|u| \leq m$ and $\varepsilon = u$. By considering problems as $\varepsilon \to 0$, singular perturbation concepts can be used to compute solutions consisting of bang-bang controls followed by singular arcs. The procedure further develops a numerical technique proposed by Jacobson, Gershwin, and Lele, as well as additional analytic methods developed by other authors.

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